Stat 155 Lecture 21 Notes

Daniel Raban

April 19, 2018

1 Examples of Shapley Value and Mechanism Design

1.1 Examples of Shapley value

Example 1.1. Consider a situation where shareholder i holds i shares for $i = 1, \dots, 4$. A deicsion needs the support of shareholders with a total of six shares:

$$\begin{aligned} v(\{1,2,3,4\}) &= v(\{1,2,3\}) = v(\{1,2,4\}) = v(\{1,3,4\}) \\ &= v(\{2,3,4\}) = v(\{2,4\}) = v(\{3,4\}) \\ &= 1, \end{aligned}$$

and $v(S) = 0$ otherwise. So we have the matrix

$$W = \begin{array}{cc} & \begin{array}{ccccccc} \{3,4\} & \{2,4\} & \{2,3,4\} & \{1,3,4\} & \{1,2,4\} & \{1,2,3\} & \{1,2,3,4\} \end{array} \\ \begin{array}{c} \{3,4\} \\ \{2,4\} \\ \{2,3,4\} \\ \{1,3,4\} \\ \{1,2,4\} \\ \{1,2,3\} \\ \{1,2,3,4\} \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \end{array},$$

with rows $S \subseteq \{1,2,3,4\}$, rows $J \subseteq \{1,2,3,4\}$, and entries $W_{S,J} = w_J(S)$. We can solve $v = Wc$ for the vector c to get

$$v(S) = w_{\{2,4\}}(S) + w_{\{3,4\}}(S) + w_{\{1,2,3\}}(S) - w_{\{2,3,4\}}(S) - w_{\{1,2,3,4\}}(S).$$

Then we get the allocation

$$\psi_1(v) = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}, \qquad \psi_2(v) = \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} = \frac{1}{4},$$

$$\psi_3(v) = \frac{1}{2} + \frac{1}{3} - \frac{1}{3} - \frac{1}{4} = \frac{1}{4}, \qquad \psi_4(v) = \frac{1}{2} + \frac{1}{2} - \frac{1}{3} - \frac{1}{4} = \frac{5}{12}.$$

Example 1.2. Players 1, 2, and 3 value a painting at a_1, a_2, and a_3 with $0 < a_1 < a_2 < a_3$. But Player 1 owns the painting, so the characteristic function is given by

$$v(\{1\}) = a_1, \qquad v(\{2\}) = v(\{3\}) = v(\{2,3\}) = 0,$$
$$v(\{1,2\}) = a_2, \qquad v(\{1,3\}) = v(\{1,2,3\}) = a_3.$$

The rational outcome, which achieves the maximal value, is for Player 3 to own the painting. What payments should occur? We can compute that

$$v(S) = a_1 w_{\{1\}}(S) + (a_2 - a_1) w_{\{1,2\}}(S) + (a_3 - a_1) w_{\{1,3\}}(S) - (a_2 - a_1) w_{\{1,2,3\}}(S),$$

so we get

$$\psi_1(v) = a_1 + (a_2 - a_1)\left(\frac{1}{2} - \frac{1}{3}\right) + (a_3 - a_1)\frac{1}{2} = \frac{2a_1 + a_2 + 3a_3}{6},$$
$$\psi_2(v) = (a_2 - a_1)\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{a_2 - a_1}{6},$$
$$\psi_3(v) = (a_3 - a_1)\frac{1}{2} - (a_2 - a_1)\frac{1}{3} = a_3 - \frac{a_2 - a_1}{6} - \frac{2a_1 + a_2 + a_3}{6}.$$

1.2 Examples of mechanism design

Example 1.3. In the women's badminton tournament in the 2012 London Olympics, there were sixteen teams, split into four groups (A, B, C, and D) of four teams each. Within a group, all pairs played a match. The top two per group advanced to a knockout tournament. In the knockout tournament, there were

1. Four quarterfinals: (i) A1 vs C2, (ii) S2 vs C1, (iii) B1 vs D2, (iv) B2 vs D1,
2. Two semifinals: Winners of (i) and (iii), and the winners of (ii) and (iv),
3. A bronze medal match between the semifinal losers,
4. A gold medal match between the semifinal winners.

However, there was an issue. There was an upset in Group D: Denmark (Pedersen/Juhl) beat the top-ranked Chinese team (Tian/Zhao), so these teams were D1 and D2, respectively. Thus, A1 would play C2, and if they won, would play Tian/Zhao (the top-ranked team) in a semifinal. But A2 would play C1 and, if they won, would play Pedersen/Juhl in a semifinal and would not play Tian/Zhao until the gold medal match.

Since there was an upset, the rank 2 and 3 teams were playing each other in a match where the winner would play the highest ranked team in the semifinals and the loser would play a lower ranked team in the semifinals. So winning the last Group A match would likely lead to a bronze medal, whereas losing it would likely lead to a silver medal. Both teams tried to lose the match, and they were both disqualified. This was a failure of tournament design, probably in the way that the rank 2 and 3 teams were both in Group A.

When we design games and mechanisms, we aim to design the rules of a game so that the outcomes have certain desired properties.

- Elections
  - Consistent with voters' rankings,
  - Fair (symmetric).
- Auctions
  - Maximize revenue for the seller,
  - Pareto efficiency,
  - Calibrated (revealing bidders' values).
- Tournaments
  - The best team is most likely to win,
  - Players have an incentive to compete.

Example 1.4. Suppose there are two candidates for president, and all voters have a preference. How do we design an election to decide between the two candidates? Voters vote; the candidate with the most votes wins. The candidate that wins is the choice of at least half of the voters. Voters never have an incentive to vote against their preferences.

Things aren't as simple with three candidates.

Example 1.5 (Condorcet's paradox[1])**.** Suppose 3 voters have the following preferences:

	1st	2nd	3rd
Voter 1	A	B	C
Voter 2	B	C	A
Voter 3	C	A	B

For every candidate, there is another candidate who is preferred by the majority. Suppose we choose candidates by a two-stage vote:

1. A vs B, then the winner vs C,
2. A vs C, then the winner vs B, or
3. B vs C, then the winner vs A.

There is no fair (symmetric) voting process that can assign a winner in these cases.

[1] Marquis de Condorcet lived in the 18th century.

Example 1.6. Here is a voting system called *plurality voting*. Voters vote for one candidate; the candidate with the most votes wins. What are the disadvantages? This system encourages strategic voting. A vote for candidates ranked third or worse is wasted. This can lead to a winner who is lowest ranked by a majority.

Example 1.7. Here is a voting system called *two-round voting* (used in France). Voters vote for one candidate. If there is not sufficient support for one candidate, a second vote is held to decide between the top-ranked two candidates.

Example 1.8. Here is a system called *contingent voting*. Voters rank the candidates. The first choices are counted. If there is not sufficient support for a single candidate, there is a second count to decide between the top-ranked two candidates. Votes supporting other candidates are distributed among the two remaining candidates according to the voters' preferences.

Example 1.9. Here is *instant-runoff voting* (used in Australia[2]). Voters rank all candidates. The number of top choices is counted. If no candidate has a majority of the top choices, the candidate with fewest top choices is eliminated and that candidate's votes are allocated to the next-ranked choices.

All of these voting systems are vulnerable to strategic voting.

Example 1.10. In a contingent voting system, say the voters have the following distribution of preferences:

	1st	2nd	3rd
30%	A	B	C
45%	B	C	A
25%	C	A	B

In this situation C is eliminated in round 1, and then A wins. What if 10% of the people in the second group lie about their preferences?

	1st	2nd	3rd
30%	A	B	C
35%	B	C	A
10%	C	B	A
25%	C	A	B

Then A is eliminated in round 1, and B wins.

What properties would we like voting methods to have? What methods possess these properties? We will discuss this next lecture.

[2] This was first used in Queensland in 1893.